

# ON THE SUBCONVEXITY ESTIMATE FOR SELF-DUAL GL(3) L-FUNCTIONS IN THE $t$ -ASPECT

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**ABSTRACT.** We improve on the subconvexity bound for self-dual GL(3)  $L$ -functions in the  $t$ -aspect. Previous results were obtained by Li and by Mckee, Sun and Ye.

## 1. INTRODUCTION

In this paper we prove a subconvexity bound for certain degree 3  $L$ -functions. Let  $\phi$  be a self-dual Hecke-Maass form for  $\mathrm{SL}(3, \mathbb{Z})$ , or equivalently, let  $\phi$  be the symmetric square lift of a Maass form for  $\mathrm{SL}(2, \mathbb{Z})$ . Then we have the following upper bound

$$L(1/2 + it, \phi) \ll_{\epsilon, \phi} t^{5/8+\epsilon}.$$

Previous results were obtained by Li [12] and more recently, by Mckee, Sun and Ye [14] who had  $\ll t^{11/16+\epsilon}$  and  $\ll t^{2/3+\epsilon}$  respectively. It is also worth mentioning Munshi's work [16], where he proved the bound  $\ll t^{11/16+\epsilon}$  on a more general setting where the forms are not necessarily self-dual.

A common feature of the results in [12], [14] and those of this paper is that they are deduced from an average result over the spectrum of the laplacian on  $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ , where  $\mathbb{H}$  is the upper half plane of complex numbers with positive imaginary part with the usual action of  $\mathrm{SL}(2, \mathbb{Z})$ . It is crucial for the result to work in the end that we have the positivity of certain  $\mathrm{GL}(3) \times \mathrm{GL}(2)$   $L$ -functions. Let  $f$  be a Hecke-Maass form for  $\mathrm{SL}(2, \mathbb{Z})$ , then we have the inequality

$$L(1/2, \phi \times f) \geq 0.$$

This result follows from the work of Lapid [10].

Munshi follows a different path where he does not need this positivity result. In fact these  $\mathrm{GL}(3) \times \mathrm{GL}(2)$   $L$ -functions do not even appear in his work.

**1.1. Statement of the main result.** We start by stating the average result from which we deduce the subconvexity bound. We refer the reader to section 2.4 for a precise definition of the  $L$ -functions involved in it.

Let  $\{f_j\}_j = \mathcal{B}$  be an orthonormal basis of Hecke-Maass forms for  $\mathrm{SL}(2, \mathbb{Z})$ , where  $f_j$  is an eigenform for the Laplacian with eigenvalue  $\frac{1}{4} + t_j^2$ .

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**Theorem 1.1.** *For every  $\epsilon > 0$ ,  $T \geq 1$  and  $\Delta \geq T^\epsilon$ , we have the inequality*

$$\sum'_{\substack{f_j \in \mathcal{B} \\ T-\Delta < t_j \leq T+\Delta}} L(1/2, \phi \times f_j) + \frac{1}{4\pi} \int_{T-\Delta}^{T+\Delta} |L(1/2 + it, \phi)|^2 dt \ll \Delta T^{5/4+\epsilon},$$

where the symbol  $\sum'$  means that we only sum over the even forms.

By taking  $\Delta = T^\epsilon$  and using the positivity of  $L(\frac{1}{2}, \phi \times f_j)$ , we deduce

**Corollary 1.2.** *We have the following bounds:*

$$L(1/2, \phi \times f_j) \ll_\phi t_j^{5/4+\epsilon} \text{ and } L(1/2 + it, \phi) \ll_\phi t^{5/8+\epsilon}.$$

In a related paper [2], Blomer proved a bound for quadratic twists of  $L$ -functions such as the ones considered in the present paper. Let  $q$  be a prime number. Suppose that  $f$  is a primitive Hecke-Maass form of level dividing  $q$  and let  $\phi$  be as above, then he proved the inequalities

$$(1) \quad L\left(\frac{1}{2}, \phi \times f \times \chi_q\right) \ll_{\phi, f} q^{5/4+\epsilon} \text{ and } L\left(\frac{1}{2} + it, \phi \times \chi_q\right) \ll_{\phi, t} q^{5/8+\epsilon},$$

where  $\chi_q$  denotes the non-trivial quadratic character of conductor  $q$ .

We remark that the exponents in Corollary 1.2 match exactly those of (1). This is not a coincidence. In fact, our method can be seen as the archimedean analog of that of [2]. The way in which we prepare the ground in order to use the archimedean version of the large sieve is inspired by ideas of Young [18], who proved the hybrid bound

$$L\left(\frac{1}{2} + it, \chi_q\right) \ll (tq)^{1/6+\epsilon}.$$

Young's method generalise the work of Conrey and Iwaniec [3] and has the nice feature that it treats the  $t$  and  $q$  aspect on the same footing.

With some considerable extra effort, we believe that the techniques in this paper should allow for proving the hybrid bound

$$L(1/2 + it, \phi \times \chi_q) \ll (tq)^{5/8+\epsilon},$$

thus improving the main result of a recent preprint by Huang [6]. In order to simplify the exposition, we decided to stick to this more restrictive case.

Finally, we would like to mention that the idea of using Young's method in this  $\mathrm{GL}(3)$  context was already present in Huang's paper. Nevertheless he did not improve on the exponent  $2/3$  by McKee, Sun and Ye. The main reason for this is that in his work (as was the case in [12] and [14]), Huang looks for upper bounds for the the sum in Theorem 1.1 that agree with the generalized Lindelöf hypothesis. This restriction forces the length of the sum to be larger. He needs  $\Delta \geq T^{1/3}$ . Thus the implied subconvexity bound is worse.

**1.2. Bounds for triple products.** Let  $\psi$  be a Maass form for  $\mathrm{SL}(2, \mathbb{Z})$ . By combining Corollary 1.2 for  $\phi = \mathrm{sym}^2 \psi$  with Ivic's bound [7]

$$L(1/2, f_j) \ll t_j^{1/3+\epsilon},$$

one gets the bound

$$L(1/2, \psi \times \psi \times f_j) \ll_{\psi} t_j^{19/12+\epsilon}$$

We remark that this estimate improves on the bound

$$L(1/2, \psi_1 \times \psi_2 \times f_j) \ll_{\Psi_1, \Psi_2} t_j^{5/3+\epsilon},$$

proved by Bernstein-Reznikov [1] in the particular case where  $\psi_1 = \psi_2$ . However, it seems that an even stronger bound (with the exponent  $4/3$ ) should follow from the methods of Suvitie [17] who proved such a bound in the analog problem for a holomorphic modular forms in the weight aspect.

**1.3. Outline of the proof.** This article belongs to a long line of papers building upon the breakthrough work of Conrey and Iwaniec [3]. An expert in the field will easily be able to recognize the many similarities and the few differences. We also recognize the great deal of influence from [2] and [18].

Our goal is to bound the sum

$$\sum_{\substack{f_j \in \mathcal{B} \\ T-\Delta \leq t_j \leq T+\Delta}} L(1/2, \phi \times f_j) + (Eis),$$

where  $(Eis)$  corresponds to the second term in Theorem 1.1, i.e. the Eisenstein contribution. We use the approximate functional equation for the Rankin-Selberg  $L$ -function  $L(1/2, \phi \times f_j)$  and we get a sum that looks like

$$\sum_{\substack{f_j \in \mathcal{B} \\ T-\Delta \leq t_j \leq T+\Delta}} \sum_{m^2 n \leq T^{3+\epsilon}} \frac{A(n, m) \lambda_j(n)}{(m^2 n)^{1/2}} + (Eis),$$

where  $(Eis.)$  stands for a similar term corresponding to the contribution from the Eisenstein series. By changing the order of summation and using the Kuznetsov formula, one gets to a sum like

$$\sum_{m^2 n \leq T^{3+\epsilon}} \frac{A(n, m)}{(m^2 n)^{1/2}} \left( \Delta T \delta_{n,1} + \sum_{\pm} \sum_{c \geq 1} \frac{1}{c} S(n, \pm 1; c) B^{\pm} \left( \frac{4\pi\sqrt{n}}{c} \right) \right),$$

where  $S(m, n; c)$  is a Kloosterman sum, and  $B^{\pm}$  is roughly the integral of a Bessel function times some other simple factors along the interval  $[T - \Delta, T + \Delta]$ . The diagonal terms are easily bounded by  $\Delta T^{1+\epsilon}$ , which is more than enough.

For the term with the plus sign (the other case is treated similarly), after separating the variables  $m$  and  $n$  by using the Hecke relations, we are faced

with the problem of estimating the sum

$$N^{-1/2} \sum_{c \geq 1} \frac{1}{c} \sum_{n \sim N} A(n, 1) S(n, 1; c) B^+ \left( \frac{4\pi\sqrt{n}}{c} \right),$$

for  $N \leq T^{3+\epsilon}$ . Now, an application of the Voronoi summation formula leads roughly to the sum

$$(2) \quad \sum_{c \geq 1} c \sum_{n \geq 1} \frac{A(1, \tilde{n})}{\tilde{n}} e \left( \pm \frac{\tilde{n}}{c} \right) \mathcal{W}^\pm \left( \frac{N\tilde{n}}{c^3}; \frac{\sqrt{N}}{c} \right),$$

where

$$\mathcal{W}^\pm(x; D) \approx x^{2/3} \int_{y \asymp 1} B^+ (4\pi D \sqrt{y}) e \left( \pm (xy)^{1/3} \right) \frac{dy}{y^{1/3}}.$$

Actually, once we use the Voronoi summation formula we are faced with a much more complicated exponential sum in place of the simple exponential factor  $e \left( \pm \frac{\tilde{n}}{c} \right)$ . A delicate study of the intervening sum was done by Blomer [2]. Although elementary his argument is rather intricate and we are glad to directly quote his calculations here.

At this point we need a stationary phase analysis of the above integral. Here we use results of [6], that were largely based on similar calculations from [18]. The final outcome of this analysis is that  $\mathcal{W}^\pm(x; D)$  is very small unless  $D \gg T$ , and in such an event, we have

$$\mathcal{W}^\pm(x; D) \approx e \left( \mp \frac{x}{D^2} \right) \times \Delta x^{1/2} \int_{-T}^T \lambda(t) \left( \frac{x}{D^2} \right)^{it} dt,$$

with  $|\lambda(t)| \leq 1$ . Once we apply these results to the sum (2), we arrive at (notice that the exponential factors cancel out!)

$$\Delta \int_{-T}^T \left| \sum_{c \ll T^{1/2}} \sum_{n \ll T^{3/2}} \frac{A(1, n)}{(cn)^{1/2}} \left( \frac{n}{c} \right)^{it} \right| dt.$$

The final touch is to use some form of the large sieve combined with the Cauchy-Schwarz inequality. By doing as such, we bound the above sum by

$$\ll \Delta T^\epsilon \left( T^{3/2} + T \right)^{1/2} \left( T^{1/2} + T \right)^{1/2} \ll \Delta T^{5/4+\epsilon},$$

as we wanted.

Notice that the fact that the size of the variables  $c$  and  $n$  are very different is somehow responsible to worsening our estimate. If we could rearrange these two variables into two other variables both with size roughly  $T$ , we would get the bound  $\Delta T^{1+\epsilon}$ , which is the limit of the method. In our case it is not clear how to employ such a trick, but if we replaced our cusp form  $\phi$  by a (maximal or minimal) parabolic Eisenstein series for  $\mathrm{SL}(3, \mathbb{Z})$ , this would be possible. In fact, the minimal parabolic case corresponds to a particular case of the result of Young [18].

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## 2. PRELIMINARIES

We begin by recalling some classical definitions and properties of Maass forms for  $\mathrm{SL}(2, \mathbb{Z})$ .

**2.1. Maass forms.** Let  $\mathbb{H}$  be the classical upper half plane and let  $\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ . Consider the spectral decomposition

$$L^2(\mathrm{SL}(2, \mathbb{Z}) \backslash \mathbb{H}) = \mathbb{C} \oplus \mathcal{C} \oplus \mathcal{E}.$$

Here,  $\mathbb{C}$  is identified with the space of constant functions on  $\mathbb{H}$ ,  $\mathcal{C}$  is the space of cusp forms and  $\mathcal{E}$  is the space of Eisenstein series.

We recall that  $\{f_j\}_{j \geq 1} = \mathcal{B}$  is a basis of Hecke-Maass cusp forms for  $\mathcal{C}$ . Each  $f_j \in \mathcal{B}$  has a Fourier decomposition

$$f_j(z) = 2y^{1/2} \sum_{n \neq 0} \rho_j(n) \sqrt{|n|} K_{it_j}(2\pi|n|y) e(nx),$$

where  $K_s$  is the classical  $K$ -Bessel function.

We can assume that all the  $f_j$  are eigenfunctions of all the Hecke operators with eigenvalues given by  $\lambda_j(n)$ . So that we have the formula

$$\rho_j(\pm n) = \rho_j(\pm 1) \lambda_j(n) n^{-1/2}.$$

The functions in  $\mathcal{C}$  can be further split in even and odd Maass forms according to whether  $f_j(-\bar{z}) = f_j(z)$  or  $f_j(-\bar{z}) = -f_j(z)$ . Finally, we define the spectral weights

$$(3) \quad \omega_j = \frac{4\pi}{\cosh(\pi t_j)} |\rho_j(1)|^2.$$

**2.2. Eisenstein series.** The Eisenstein series  $E(z, s)$  has a Fourier decomposition as follows:

$$(4) \quad E(z, s) = y^s + \eta(s) y^s + 2y^{1/2} \sum_{n \neq 0} \eta(n, s) \sqrt{|n|} K_{s-1/2}(2\pi|n|y) e(nx).$$

The Fourier coefficients can be given explicitly [3, p. 1187-1188] by

$$\eta(s) = \pi^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)},$$

and

$$\eta(n, s) = \pi^s \Gamma(s)^{-1} \zeta(2s)^{-1} |n|^{-1/2} \sigma_{s-1/2}(|n|),$$

where for  $n > 0$ ,  $s \in \mathbb{C}$ ,

$$\sigma_s(n) = \sum_{ad=n} \left( \frac{a}{d} \right)^s.$$

We also define the spectral weights

$$(5) \quad \omega(t) = \frac{4\pi}{\cosh(\pi t)} |\eta(1, 1/2 + it)|^2.$$

**2.3. Kuznetsov formula.** In what follows we recall the Kuznetsov formula which is a key ingredient of our proof. The next lemma is exactly [3, Eq. (3.17)].

**Lemma 2.1.** *For  $m, n \geq 1$ ,  $(mn, q) = 1$  and any even test function  $h$  satisfying the following conditions*

- (i)  *$h$  is holomorphic in  $|\operatorname{Im}(t)| \leq \frac{1}{2} + \epsilon$ ,*
- (ii)  *$h(t) \ll (1 + |t|)^{-2-\epsilon}$  in the above strip,*

*we have the following identity:*

$$\begin{aligned} \sum'_{f_j \in \mathcal{B}} h(t_j) \omega_j \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(t) \omega(t) \sigma_{it}(m) \sigma_{it}(n) dt \\ = \frac{1}{2} \delta_{m,n} D + \frac{1}{2} \sum_{\pm} \sum_{c \equiv 0 \pmod{c}} \frac{S(m, \pm n; c)}{c} B^{\pm} \left( \frac{4\pi \sqrt{mn}}{c} \right), \end{aligned}$$

where  $\mathcal{B}$  is a basis of Hecke-Maass forms, and  $\sum'$  means that we only sum over the even forms,  $\delta_{m,n}$  is the Kronecker symbol, and

$$(6) \quad \begin{cases} D = \frac{2}{\pi} \int_0^{+\infty} h(t) \tanh(\pi t) t dt, \\ B^+(x) = 2i \int_{-\infty}^{+\infty} J_{2it}(x) \frac{h(t)t}{\cosh(\pi t)} dt, \\ B^-(x) = \frac{4}{\pi} \int_{-\infty}^{+\infty} K_{2it}(x) \sinh(t) h(t) t dt, \end{cases}$$

where  $J_\nu$  and  $K_\nu$  are the standard  $J$  and  $K$  Bessel functions respectively.

**2.4. GL(3) Maass forms and  $L$ -functions.** Let  $\phi$  be a Hecke-Maass form for  $\operatorname{SL}(3, \mathbb{Z})$  of type  $(\nu_1, \nu_2)$  whose Hecke eigenvalues are  $A(n, m)$ . We refer the reader to Goldfeld's book [5] for a precise definition and further information about these forms. In this paper we only use the  $L$ -functions associated to these forms, so our attention will be focused on them.

We consider the  $L$ -function associated to  $\psi$ , given by

$$L(s, \phi) := \sum_{n \geq 1} \frac{A(n, 1)}{n^s},$$

and for a GL(2) Hecke-Maass cusp form  $f$  with Hecke eigenvalues  $\lambda(n)$ , we consider the  $L$ -function of the Rankin-Selberg convolution of  $\phi$  and  $f$ , i.e.

$$(7) \quad L(s, \phi \times f) = \sum_{n \geq 1} \frac{A(n, m) \lambda(n)}{(m^2 n)^s}.$$

2.4.1. *On the coefficients  $A(n, m)$ .* At some point in our proof, it will be advantageous to separate the variables  $m$  and  $n$  in  $A(n, m)$  and this will be done by means of the Hecke relations. The following is obtained by applying Möbius inversion to [5, Theorem 6.4.11]:

$$(8) \quad A(n, m) = \sum_{d|(m, n)} \mu(d) A\left(\frac{n}{d}, 1\right) A\left(1, \frac{m}{d}\right).$$

We shall also need estimate for the coefficients  $A(n, m)$ . For this we restrict ourselves to the case where  $\phi$  is a symmetric square lift of a GL(2) Hecke Maass form. In this case we have both the pointwise bound

$$A(n, m) \ll (mn)^{7/32+\epsilon},$$

and the average result

$$(9) \quad \sum_{m \leq X} A(1, m)^2 \ll X.$$

The first one is a consequence of the fact that  $\phi$  comes from the symmetric square lift of GL(2) form and the Kim-Sarnak bound, while the second comes from the Rankin-Selberg theory. Combining these estimates with the multiplicativity property of the  $A(n, m)$ , i.e.

$$A(n_1 n_2, m_1 m_2) = A(n_1, m_1) A(n_2, m_2), \text{ if } (m_1 n_1, m_2 n_2) = 1,$$

we obtain the upper bound:

$$(10) \quad \sum_{m \leq X} |A(a, bm)|^2 \ll X(ab)^{\frac{7}{16}+\epsilon}.$$

2.4.2. *Approximate functional equation.* It is common knowledge among specialists that the value of an  $L$ -function on the critical line can be expressed by an essentially finite sum. The next two lemmas are a consequence of [9, Theorem 5.3]. In the following we consider the Langlands parameters

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \quad \alpha_2 = -\nu_1 + \nu_2, \quad \alpha_3 = -2\nu_1 + \nu_2 - 1.$$

**Lemma 2.2.** *We have the following identities:*

$$L(1/2, \phi \times f_j \times \chi) = 2 \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{A(n, m) \lambda_j(n) \chi(n)}{(m^2 n)^{1/2}} V_{t_j} \left( \frac{m^2 n}{q^2} \right),$$

and

$$|L(1/2 + it, \phi \times \chi)|^2 = 2 \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{A(n, m) \sigma_{it}(n) \chi(n)}{(m^2 n)^{1/2}} V_t \left( \frac{m^2 n}{q^2} \right),$$

where

$$V_t(y) = \frac{1}{2\pi i} \int_{(3)} (\pi^2 y)^{-u} \prod_{\pm} \prod_{i=1}^3 \frac{\Gamma\left(\frac{1/2 \pm it + u - \alpha_i}{2}\right)}{\Gamma\left(\frac{1/2 \pm it - \alpha_i}{2}\right)} e^{u^2} \frac{du}{u}.$$

The next lemma shows that all the sums in Lemma 2.2 are essentially bounded. It also describes explicitly the dependency of  $V_t$  on the variable  $t$ .

**Lemma 2.3.** (i) For  $k \geq 0$

$$y^k V_t^{(k)}(y) \ll \left(1 + \frac{y}{(1+|t|)^3}\right)^{-A},$$

and

$$y^k V_t^{(k)}(y) = \delta_k + O\left(\left(\frac{y}{(1+|t|)^3}\right)^\alpha\right),$$

for any  $0 < \alpha \leq \frac{1}{3} \min_{1 \leq i \leq 3} (1/2 - |\Re(\alpha_i)|)$ , where  $\delta_0 = 1$  and  $\delta_k = 0$  otherwise.

(ii) For  $1 < L \ll T^\epsilon$ ,  $\epsilon > 0$ , and  $|t - T| \ll T^{1-2\epsilon}$ , we have the following approximation

$$\begin{aligned} V_t(y) = \sum_{k=0}^{K/2} \sum_{\ell=0}^{K/2} t^{-2k} \left(\frac{t^2 - T^2}{T^2}\right)^\ell V_{k,\ell}\left(\frac{y}{T^3}\right) + O(y^{-\epsilon}(1+|T|)^\epsilon e^{-L}) \\ + O\left(\left(\frac{1+|t-T|}{T}\right)^{K+1} \left(1 + \frac{y}{T^3}\right)^{-A}\right), \end{aligned}$$

where

$$(11) \quad V_{k,\ell}(y) = \frac{1}{2\pi i} \int_{\epsilon-iL}^{\epsilon+iL} P_{k,\ell}(u) (2\pi)^{-3u} y^{-u} e^{u^2} \frac{du}{u}.$$

*Proof.* For a proof, one can check [6, Lemma 2.2].  $\square$

We shall use this result with  $U = (\log T)^2$ , so that the first error term is  $O_A(T^{-A})$  for every  $A > 0$ .

**Remark 2.1.** Thanks to the works of Luo-Rudnick-Sarnak [13] on the Ramanujan conjecture, one can take any  $\alpha \leq \frac{1}{10}$  in the lemma above.

**2.5. Voronoi summation formula.** In the next lemma we recall the Voronoi summation formula for  $\mathrm{SL}(3, \mathbb{Z})$ -Maass forms. The Voronoi formula is a generalization of the classical Poisson summation formula and is in a certain sense equivalent to the functional equation for  $L(s, \phi \times \chi)$  where  $\chi$  is a multiplicative character. The first version of this formula was obtained by Miller and Schmid [15]. The version we give here is [2, Lemma 3]

**Lemma 2.4.** Let  $w : (0, +\infty) \rightarrow \mathbb{C}$  be a smooth function with compact support. Let  $\widehat{w}(s)$  denote its Melin transform and let

$$G^\pm(x) := \prod_{i=1}^3 \frac{\Gamma\left(\frac{s+\alpha_i}{2}\right)}{\Gamma\left(\frac{1-s-\alpha_i}{2}\right)} \pm \frac{1}{i} \prod_{i=1}^3 \frac{\Gamma\left(\frac{1+s+\alpha_i}{2}\right)}{\Gamma\left(\frac{2-s-\alpha_i}{2}\right)}.$$



Then we have

$$(12) \quad \sum_{n=1}^{+\infty} A(n, m) e\left(\frac{\bar{d}n}{c}\right) w(n) \\ = \frac{\pi^{3/2}}{2} c \sum_{\pm} \sum_{n_1 | cm} \sum_{n_2=1}^{+\infty} \frac{A(n_1, n_2)}{n_1 n_2} S\left(dm, \pm n_2; \frac{mc}{n_1}\right) \mathcal{W}\left(\frac{n_1^2 n_2}{c^3 m}\right),$$

where

$$(13) \quad \mathcal{W}^{\pm}(x) = \frac{x}{2\pi i} \int_{(3)} G^{\pm}(s) \widehat{w}(y) dy.$$

### 3. INITIAL STEPS FOR THEOREM 1.1

We want to prove

$$\sum_{\substack{f_j \in \mathcal{B} \\ T-\Delta \leq t_j \leq T+\Delta}} L(1/2, \phi \times f_j) + \frac{1}{4\pi} \int_{T-\Delta}^{T+\Delta} |L(1/2 + it, \phi)|^2 dt \ll \Delta T^{5/4+\epsilon}.$$

We shall consider the spectrally normalized forms

$$\mathcal{M} := \sum'_{\substack{f_j \in \mathcal{B} \\ T-\Delta \leq t_j \leq T+\Delta}} \omega_j L\left(\frac{1}{2}, \phi \times f_j\right) + \frac{1}{4\pi} \int_{T-\Delta}^{T+\Delta} \omega(t) \left| L\left(\frac{1}{2} + it, \phi\right) \right|^2 dt,$$

where  $\omega_j$  and  $\omega(t)$  are given by (3) and (5) respectively. There is not such a big loss because we have the inequalities

$$\omega_j \gg t_j^{-\epsilon}, \quad \omega(t) \gg t^{-\epsilon}.$$

The first of these upper bounds is contained in [8, Theorem 8.3] and the second one is a classical result.

Hence we need to prove

$$(14) \quad \mathcal{M} \ll \Delta T^{5/4+\epsilon}.$$

We want to estimate  $\mathcal{M}$  by means of the Kuznetsov formula. To do so, we need to consider a smooth variant of  $\mathcal{M}$ . Precisely, we let

$$(15) \quad h(t) = \frac{1}{\cosh\left(\frac{t-T}{\Delta}\right)} + \frac{1}{\cosh\left(\frac{t+T}{\Delta}\right)}.$$

Here one could take other nice holomorphic even functions such that  $h(t) \gg 1$  in the region  $|t - T| \leq \Delta$  but we take this particular one so that we can

directly quote results from [18] and [6]. Since  $h(t) \gg 1$  in  $[T - \Delta, T + \Delta]$ , it follows that

$$\mathcal{M} \ll \sum'_{f_j \in \mathcal{B}} h(t_j) \omega_j L \left( \frac{1}{2}, \phi \times f_j \right) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(t) \omega(t) \left| L \left( \frac{1}{2} + it, \phi \right) \right|^2 dt.$$

Applying the approximate functional equation (see Lemma 2.2), the right-hand side above equals

$$\begin{aligned} \sum'_{f_j \in \mathcal{B}} h(t_j) \omega_j \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{A(n, m) \lambda_j(n)}{(m^2 n)^{1/2}} V_{t_j}(m^2 n) \\ + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h(t) \omega(t) \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{A(n, m) \sigma_{it}(n)}{(m^2 n)^{1/2}} V_t(m^2 n) dt, \end{aligned}$$

By Lemma 2.3, there exists  $K > 0$  for which is enough to bound

$$(16) \quad \mathcal{M}^\dagger := \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{A(n, m)}{(m^2 n)^{1/2+u}} V \left( \frac{m^2 n}{T^3} \right) \times \left( \sum'_{f_j \in \mathcal{B}} h_{k,\ell}(t_j) \omega_j \lambda_j(n) + \frac{1}{4\pi} \int_{-\infty}^{+\infty} h_{k,\ell}(t) \omega(t) \sigma_{it}(n) dt \right),$$

uniformly in  $u \in [\epsilon - i(\log T)^2, \epsilon + i(\log T)^2]$  and  $0 \leq k, \ell \leq K$ , where  $V = V_{k,\ell}$  is of the form (11), and

$$(17) \quad h_{k,\ell} = t^{-2k} T^{-2\ell} (t^2 - T^2)^\ell h(t).$$

In the following we will give all the details only in the case where  $k = \ell = 0$ . The other cases can be handled similarly and amount to smaller order terms.

We are now in a perfect position to use the Kuznetsov formula to the terms between parenthesis in (16). We thus obtain

$$\begin{aligned} \mathcal{M}^\dagger = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n, m) \chi(n)}{(m^2 n)^{1/2+u}} V \left( \frac{m^2 n}{T^3} \right) \left( \frac{1}{2} \delta_{n,1} D + \right. \\ \left. \frac{1}{2} \sum_{c \geq 1} \frac{1}{c} \sum_{\pm} S(n, \pm 1; c) B^\pm \left( \frac{4\pi \sqrt{n}}{c} \right) \right), \end{aligned}$$

where  $D$  and  $B^\pm(x)$  are as in (6).

It follows from (15) that  $D \ll \Delta T^{1+\epsilon}$  and hence, by Cauchy-Schwarz and (9), we have

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n, m)}{(m^2 n)^{1/2+u}} V\left(\frac{m^2 n}{T^3}\right) \delta_{n,1} D \\ \ll \Delta T^{1+\epsilon} \sum_{m \ll T^{3/2+\epsilon}} \frac{1}{m} \ll \Delta T^{1+\epsilon}, \end{aligned}$$

which suffices for our purposes. Now we need to estimate the off-diagonal terms. Let

$$S_{\sigma} := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n, m)}{(m^2 n)^{1/2+u}} V\left(\frac{m^2 n}{T^3}\right) \sum_{c \equiv 0 \pmod{q}} \frac{1}{c} S(n, \sigma; c) B^{\sigma}\left(\frac{4\pi\sqrt{n}}{c}\right),$$

for  $\sigma = \pm$ . In the following, we must separate the variables  $m$  and  $n$  in  $A(n, m)$  by means of the Hecke relations (see (8)). This together with a change of variables and a dyadic decomposition on the  $n$  variable shows that

$$S_{\sigma} \ll \sum_{\delta^3 m^2 \leq T^{3+\epsilon}} \sum \frac{|A(1, m)|}{\delta^{3/2} m} \sup_{N \ll \frac{T^{3+\epsilon}}{\delta^3 m^2}} \frac{|S_{\sigma}(N; \delta)|}{N^{1/2}} + O(T^{-A}),$$

where

$$(18) \quad S_{\sigma}(N; \delta) = \sum_{c=1}^{+\infty} \frac{1}{c} \sum_n A(n, 1) S(\delta n, \sigma; c) w_{\sigma}\left(\frac{n}{N}; \frac{\sqrt{\delta N}}{c}\right),$$

where

$$w_{\sigma}(y; D) := w(y) y^{-u} B^{\sigma}(4\pi D \sqrt{y}),$$

for some smooth function  $w$  with compact support.

We must prove the following

**Proposition 3.1.** *Let  $S_{\sigma}(N; \delta)$  be as above. Then we have the inequality*

$$S_{\sigma}(N; \delta) \ll \Delta \delta^{1/2} N^{1/2} T^{5/4+\epsilon},$$

*uniformly for  $\delta^3 N \ll T^{3+\epsilon}$*

**3.1. Applying the Voronoi formula.** The next step is to apply the Voronoi summation formula on the  $n$  variable. Opening the Kloosterman sum and using Lemma 2.4, we obtain

$$(19) \quad S_{\sigma}(N; \delta) = \frac{\pi^{3/2}}{2} \sum_{\pm} \sum_{c=1}^{+\infty} \frac{1}{\delta_0} \sum_{n_1 | c_1} \sum_{n_2=1}^{\infty} \frac{A(n_1, n_2)}{n_1 n_2} \\ \times \mathcal{W}_{\sigma}^{\pm} \left( \frac{N n_1^2 n_2}{c_1^3}; \frac{\sqrt{\delta N}}{c} \right) \mathcal{T}_{\delta, n_1, n_2}^{\pm, \sigma}(c),$$

where  $\delta_0 = (\delta, c)$ ,  $\delta' = \delta/\delta_0$ ,  $c' = c/\delta_0$ ,

$$\mathcal{T}_{\delta, n_1, n_2}^{\pm, \sigma}(c) := \sum_{d \pmod{c}}^* e\left(\frac{\sigma d}{c}\right) S(\bar{\sigma}' d, \pm n_2; c'/n_1),$$

and  $\mathcal{W}_\sigma^\pm(x; D)$  is defined as in (13) with  $w(\cdot)$  replaced by  $w_\sigma(\cdot, D)$ .

In the next sections we shall study the integral transform  $\mathcal{W}_\sigma^\pm(\cdot, \cdot)$  and the exponential sum  $\mathcal{T}_{\delta, n_1, n_2}^{\pm, \sigma}(c)$ .

#### 4. STATIONARY PHASE METHOD

In this section we deduce a particularly nice formula for  $\mathcal{W}^\pm$ . Precisely, we show that after multiplying it by a suitable oscillating factor, we can write it roughly as an integral of length  $T$  of an archimedean character, which will be perfect for the application of the large sieve later on.

The next lemma gives the asymptotic behavior of the integral transform  $\mathcal{W}^\pm$ . This can be found in some form in [11], but here we use a slightly more precise form given by Blomer (see [2, Lemma 6]).

**Lemma 4.1.** *Let  $K > 0$ . There exist constants  $\gamma_\ell$  depending only on the Langlands parameters  $\alpha_i$  of  $\phi$  such that for any compactly supported function  $w$  and  $x \geq 1$ , we have*

$$\mathcal{W}^\pm(x) = x \int_0^{+\infty} w(y) \sum_{j=1}^K \frac{\gamma_\ell}{(xy)^{\ell/3}} e\left(\pm(xy)^{1/3}\right) dy + O(x^{1-K/3}),$$

where the implied constant depend only on the  $\alpha_i$ ,  $\|w\|_\infty$  and  $K$ .

The next lemma combines Lemma 4.1 with stationary phase arguments due to Young (see [18, Lemma 8.1]). The details were carried out by Huang (see [6, Lemma 4.4]).

**Lemma 4.2.** *Let  $x \gg T^{-B}$  for some large but fixed  $B$ . Let*

$$(20) \quad \widetilde{\mathcal{W}}_\sigma^\pm(x; D) := e\left(\mp \sigma \frac{x}{D^2}\right) \mathcal{W}(x; D).$$

We have  $\widetilde{\mathcal{W}}_\sigma^\pm(x; D) \ll T^{-A}$ , unless

$$(21) \quad \begin{cases} D \gg T \Delta^{1-\epsilon}, & \text{if } \sigma = 1, \\ D \asymp T, & \text{if } \sigma = -1. \end{cases}$$

If  $x \gg T^\epsilon$ , we have

$$(22) \quad \widetilde{\mathcal{W}}_\sigma^\pm(x; D) = \Delta x^{5/6} \sum_{j=1}^K \frac{\gamma_\ell}{x^{\ell/3}} L_j(x; D) + O_A(T^{-A}),$$

where  $L_j$  is a function that takes the form

$$L_j(x; D) = \int_{|t| \ll U} \lambda_{X, D}(t) \left(\frac{x}{D^2}\right)^{it} dt,$$

with the following parameters. Here  $\lambda_{X,T}(t) \ll 1$  does not depend on  $x$  or  $D$ . If  $\sigma = 1$ , then  $U = T^2/D$  and  $L_j$  vanishes unless

$$X \asymp D^3, \text{ and } D \gg \Delta T^{1-\epsilon}.$$

If  $\sigma = -1$ , then  $U = T^{2/3}X^{1/3}D^{-2/3}$  and  $L_j$  vanishes unless

$$X \ll D^3 \Delta^{-3+\epsilon}, \text{ and } D \asymp T.$$

### 5. TREATMENT OF THE EXPONENTIAL SUM $\mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c)$

Before we use the results of the last section, we must first write a formula for  $\mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c)$  that has the dependence in  $n_2$  in a rather nice explicit way. In particular, we will find the term  $e\left(\mp\sigma\frac{\delta_0 n_1^2 n_2}{\delta' c'}\right)$  necessary to form  $\widetilde{\mathcal{W}}^\pm(\dots)$  (see (22)).

This is a very delicate calculation and while doing it many new variables will arise. But almost all of them play no important role in the argument.

We remark that our definition of  $\mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c)$  can be seen as a special case of  $\mathcal{T}_{\delta,c_1,n_1,n_2}^{\pm,\sigma}(c,q)$  in [2, p. 1407]. Indeed we have

$$\mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c) = \frac{1}{c} \mathcal{T}_{\delta, \frac{c}{(\delta,c)}, n_1, n_2}^{\pm,\sigma}(c, 1).$$

Thus, [2, Lemma 12] becomes

**Lemma 5.1.** *Let  $\delta_0 = (\delta, c)$ ,  $\delta' = \delta/\delta_0$  and  $c' = c/\delta_0$ .*

*If  $(\delta_0, c') = 1$ , then*

$$(23) \quad \mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c) = e\left(\mp\sigma\frac{\overline{\delta'}\delta_0 n_1^2 n_2}{c'}\right) \frac{\varphi(c'/n_1)}{\varphi(c')} \frac{\mu(\delta_0)}{\delta_0} c \\ \times \sum_{\substack{d'_2 f_1 f_2 = c' \\ (d'_2, f_1 n_1 n_2) = 1 \\ (f_1, f_2) = 1, f_2 | n_1}} \frac{\mu(f_1)^2 \mu(f_2)}{f_1} e\left(\pm\sigma\frac{\overline{\delta'} d'_2 \delta_0 f_2 (n'_1)^2 n_2}{f_1}\right),$$

where  $n'_1 = n_1/f_2$  and  $\mathcal{T}_{\delta,n_1,n_2}^{\pm,\sigma}(c) = 0$  otherwise.

By a double application of the classical formula  $e(\bar{a}/b + \bar{b}/a) = e(1/ab)$ , we see that the product of the two exponentials on the right-hand side of (23) equals

$$e\left(\mp\sigma\frac{\delta_0 n_1^2 n_2}{\delta' c'}\right) e\left(\pm\sigma\frac{\overline{d'_2} \delta_0 f_2 (n'_1)^2 n_2}{\delta' f_1}\right).$$

The first factor combines perfectly with  $\mathcal{W}_\sigma^\pm\left(\frac{N n_1^2 n_2}{c_1^3}; \frac{\sqrt{\delta N}}{c}\right)$  to form  $\widetilde{\mathcal{W}}_\sigma^\pm\left(\frac{N n_1^2 n_2}{c_1^3}; \frac{\sqrt{\delta N}}{c}\right)$ .

We conclude that

$$\begin{aligned}
S_\sigma(N; \delta) &= \frac{\pi^{3/2}}{2} \sum_{\pm} \sum_{\delta_0 \delta' = \delta} \frac{\mu(\delta_0)}{\delta_0} \sum_{\substack{d'_2, f_1, f_2 \\ (\delta, d'_2 f_1 f_2) = 1 \\ (f_1, f_2) = 1, (d'_2, f_1 f_2) = 1}} \sum_{\substack{n_1 | f_1 \\ (d'_2, n'_1) = 1}} \\
&\times \sum_{\substack{n_2 \\ (d'_2, n_2) = 1}} \mu(f_1)^2 \mu(f_2) d'_2 \frac{\varphi(d'_2 f_1 / n_1)}{\varphi(d'_2 f_1 f_2)} \frac{A(f_2 n_1, n_2)}{n'_1 n_2} \\
&\times e \left( \pm \sigma \frac{\overline{d'_2} \delta_0 f_2 (n'_1)^2 n_2}{\delta' f_1} \right) \widetilde{\mathcal{W}}_\sigma^\pm \left( \frac{N(n'_1)^2 n_2}{(d'_2 f_1)^3 f_2}; \frac{\sqrt{\delta N}}{\delta_0 d'_2 f_1 f_2} \right).
\end{aligned}$$

We write  $f_1 = gn'_1$ . Then we have

$$\begin{aligned}
S_\sigma(N; \delta) &= \frac{\pi^{3/2}}{2} \sum_{\pm} \sum_{\delta_0 \delta' = \delta} \frac{\mu(\delta_0)}{\delta_0} \sum_{\substack{d'_2, f_2, g, n'_1 \\ (\delta, d'_2 f_2 g n'_1) = 1 \\ (f_2, g n'_1) = 1 \\ (d'_2, f_2 g n'_1) = 1}} \sum_{\substack{n_2 \\ (d'_2, n_2) = 1}} \frac{\mu(g n'_1)^2 \mu(f_2) d'_2}{\varphi(f_2 n'_1) n'_1} \\
&\times \frac{A(f_2 n'_1, n_2)}{n_2} e \left( \pm \sigma \frac{\overline{d'_2} \delta_0 f_2 n'_1 n_2}{\delta' g} \right) \widetilde{\mathcal{W}}_\sigma^\pm \left( \frac{N n_2}{(d'_2 g)^3 f_2 n'_1}; \frac{\sqrt{\delta N}}{\delta_0 d'_2 f_2 g n'_1} \right).
\end{aligned}$$

Since  $(\delta_0 d'_2 f_2 n'_1, \delta' g) = 1$ . Let

$$s = (\delta' g, n_2), \quad n_2 = n'_2 s, \quad (n'_2, \delta' g/s).$$

We deduce that

$$\begin{aligned}
(24) \quad S_\sigma(N; \delta) &= \frac{\pi^{3/2}}{2} \sum_{\pm} \sum_{\delta_0 \delta' = \delta} \frac{\mu(\delta_0)}{\delta_0} \sum_{\substack{f_2, g, n'_1 \\ (\delta, f_2 g n'_1) = 1 \\ (f_2, g n'_1) = 1}} \frac{\mu(g n'_1)^2 \mu(f_2)}{\varphi(f_2 n'_1) n'_1} \sum_{s | \delta' g} \frac{1}{s} \\
&\sum_{\substack{d'_2 \\ (d'_2, \delta f_2 g n'_1) = 1}} \sum_{\substack{n'_2 \\ (n'_2, d'_2 \delta' g/s) = 1}} \frac{A(f_2 n'_1, n'_2 s) d'_2}{n'_2} e \left( \pm \sigma \frac{\overline{d'_2} \delta_0 f_2 n'_1 n'_2}{\delta' g/s} \right) \\
&\times \widetilde{\mathcal{W}}_\sigma^\pm \left( \frac{N n'_2}{(d'_2 g)^3 f_2 n'_1}; \frac{\sqrt{\delta N}}{\delta_0 d'_2 f_2 g n'_1} \right).
\end{aligned}$$

Let

$$x := \frac{N n_2}{(d'_2 g)^3 f_2 n'_1}, \quad \text{and} \quad D := \frac{\sqrt{\delta N}}{\delta_0 d'_2 f_2 g n'_1}.$$

We note that by Lemma 4.2, the contribution to the right-hand side of (24) of the terms where

$$D \ll T^{1-\epsilon}.$$

is negligible. This implies that we can impose  $d'_2 \ll T^\epsilon \frac{\sqrt{\delta N}}{T \delta_0 f_2 g n'_1}$ . And hence

$$x \gg T^{3-\epsilon} \frac{\delta_0^3 (f_2 n'_1)^2 n'_2}{(\delta^3 N)^{1/2}} \gg T^{\frac{3}{2}-\epsilon},$$

since we are assuming  $\delta^3 N \leq T^{3+\epsilon}$ . That means that we can apply the second part of Lemma 4.2 to the remaining terms in the right-hand side of (24). Upon making a dyadic decomposition of the variables  $d'_2$  and  $n'_2$ , we deduce that except for a negligible term ( $O(T^{-A})$ ),  $S_\sigma(N; \delta)$  is

$$\begin{aligned}
(25) \quad & \ll \Delta T^\epsilon \sum_{\pm} \sum_{\delta_0 \delta' = \delta} \frac{1}{\delta_0} \sum_{\substack{f_2, g, n'_1, r \\ (\delta, f_2 g n'_1) = 1 \\ (f_2, g n'_1) = 1 \\ (r, \delta f_2 g n'_1)}} \sum_{s | \delta' g} \frac{1}{(f_2 g)^{3/2} (n'_1)^{5/2} r s^{1/2}} \\
& \times \sup_{D_2, N_2} \frac{N^{1/2}}{D_2^{1/2} N_2^{1/2}} \int_{|t| \ll U} \left| \sum_{\substack{D_2 < d'_2 \leq D_2 \\ (d'_2, \delta f_2 g n'_1) = 1}} \sum_{\substack{N_2 < n'_2 \leq 2N_2 \\ (n'_2, d'_2 \delta' g/s) = 1}} \alpha(d'_2) \beta(n'_2) \right. \\
& \left. \times A(f_2 n'_1, n'_2 r s) e \left( \pm \sigma \frac{\overline{d'_2} \delta_0 f_2 n'_1 n'_2}{\delta' g/s} \right) \left( \frac{n'_2}{d'_2} \right)^{it} \right| dt,
\end{aligned}$$

where

$$(26) \quad \begin{cases} 1 \ll D_2 \ll \frac{(\delta N)^{1/2}}{T \delta_0 f_2 g n'_1 r}, \\ 1 \ll N_2 \ll \frac{(\delta^3 N)^{1/2}}{\delta_0^3 (f_2 n'_1)^2 r s}, \\ U \ll T. \end{cases}$$

Note that in particular, we have

$$(27) \quad r \ll \frac{(\delta N)^{1/2}}{T} \ll T^{1/2+\epsilon}.$$

## 6. THE LARGE SIEVE AND END OF THE PROOF

The final step consists in applying some version of the large sieve to the integral in the right-hand side of (25), but before we do so we need to decompose the exponential factor as a combination of characters weighted by Gauss sums. All this is accomplished in the following lemma. We have

**Lemma 6.1.** *Let  $a, b, c \in \mathbb{Z}$  such that  $(ab, c) = 1$ . Let  $D, N \geq 1$ . Suppose  $\alpha_d, \beta_n \in \mathbb{C}$  and  $U \geq 1$ . Then for any  $\epsilon > 0$ , we have*

$$\begin{aligned}
& \int_{-U}^U \left| \sum_{\substack{d \sim D \\ (dn, c) = 1}} \sum_{n \sim N} \alpha_d \beta_n e \left( \frac{\overline{a} d b n}{c} \right) \left( \frac{n}{d} \right)^{it} \right| dt \\
& \ll c^{1/2} (D + U)^{1/2} (N + U)^{1/2} \|\alpha\|_2 \|\beta\|_2,
\end{aligned}$$

where

$$\|\alpha\|_2 := \left( \sum_d |\alpha_d|^2 \right)^{1/2}, \quad \|\beta\|_2 := \left( \sum_n |\beta_n|^2 \right)^{1/2}.$$

*Proof.* For every multiplicative character modulo  $c$ , we let

$$\tau(\chi) := \sum_{x \pmod{c}} \chi(x) e\left(\frac{x}{c}\right)$$

be the Gauss sum associated to  $\chi$ . Then by orthogonality of characters, the integral on the left-hand side is

$$\ll \frac{1}{\varphi(c)} \sum_{\chi \pmod{c}} |\tau(\chi)| \int_{-U}^U \left| \sum_{\substack{d \sim D \\ (d,c)=1}} \sum_{n \sim N} \alpha_d \beta_n \chi(d/n) \left(\frac{n}{d}\right)^{it} \right| dt.$$

By using the Cauchy-Schwarz inequality we see that the inner integral is bounded by

$$\left( \int_{-U}^U \left| \sum_{\substack{d \sim D \\ (d,c)=1}} \alpha_d \chi(d) d^{it} \right|^2 dt \right)^{1/2} \left( \int_{-U}^U \left| \sum_{\substack{n \sim N \\ (n,c)=1}} \beta_n \bar{\chi}(n) n^{it} \right|^2 dt \right)^{1/2}.$$

Now the large sieve (see [4, Theorem 2]) implies that the above expression is

$$\ll (D+U)^{1/2} (N+U)^{1/2} \|\alpha\|_2 \|\beta\|_2.$$

The Lemma now follows from the classical bound for the Gauss sums.  $\square$

Using Lemma 6.1 for the integral in (25), we may bound it by

$$\left( \frac{\delta' g}{s} \right)^{1/2} D_2^{1/2} (D_2 + U)^{1/2} (N + U)^{1/2} \left( \sum_{n_2'' \sim N_2} A(f_2 n_1', n_2'' r s)^2 \right)^{1/2}.$$

We estimate the sum over  $n_2''$  by means of the Rankin-Selberg bound (see (10)) and we recall that because of (26), and the inequality  $\delta^3 N \leq T^{3+\epsilon}$ , we have

$$D_2 \ll T^{1/2+\epsilon}/r, \quad N_2 \ll T^{3/2+\epsilon}/r, \quad U \ll T.$$

Putting everything together and applying it to (25), we get (recall (27))

$$\frac{S_\sigma(N; \delta)}{N^{1/2}} \ll T^\epsilon \Delta \delta^{1/2} \sum_{r \ll T^{1/2+\epsilon}} r^{-25/32} \left( \frac{T^{1/2}}{r} + T \right)^{1/2} \left( \frac{T^{3/2}}{r} + T \right)^{1/2},$$

since all the other sums are convergent. From this we deduce



$$\begin{aligned} \frac{S_\sigma(N; \delta)}{N^{1/2}} &\ll \Delta T^\epsilon \sum_{r \ll T^{1/2}} \delta^{1/2} r^{-25/32} \left( \frac{T^{5/4}}{r^{1/2}} + T \right) \\ &\ll \Delta \delta^{1/2} \left( T^{5/4+\epsilon} + T^{39/32+\epsilon} \right). \end{aligned}$$

We have thus proved Proposition 3.1 and hence Theorem 1.1.

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